The Eulerian Distribution on Involutions is Indeed Unimodal

Victor J. W. Guo, Jiang Zeng

The Eulerian Distribution on Involutions is Indeed Unimodal

Victor J. W. Guo[1] and Jiang Zeng[2]

Institut Camille Jordan, Université Claude Bernard (Lyon I), F-69622, Villeurbanne Cedex, France

Abstract. Let $I_{n,k}$ (respectively, $J_{n,k}$) be the number of involutions (respectively, fixed-point free involutions) of $\{1,\ldots,n\}$ with k descents. Motivated by Brenti's conjecture which states that the sequence $I_{n,0}, I_{n,1}, \ldots, I_{n,n-1}$ is log-concave, we prove that the two sequences $I_{n,k}$ and $J_{2n,k}$ are unimodal in k, for all n. Furthermore, we conjecture that there are nonnegative integers $a_{n,k}$ such that

$$\sum_{k=0}^{n-1} I_{n,k} t^k = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k} t^k (1+t)^{n-2k-1}.$$

This statement is stronger than the unimodality of $I_{n,k}$ but is also interesting in its own right.

Keywords: Involutions, Descent number, Unimodality, Eulerian polynomial, Zeilberger's algorithm

AMS Subject Classifications (2000): Primary 05A15; Secondary 05A20

1 Introduction

A sequence $a_0, a_1, \ldots, a_n$ of real numbers is said to be *unimodal* if for some $0 \le j \le n$ we have $a_0 \le a_1 \le \cdots \le a_j \ge a_{j+1} \ge \cdots \ge a_n$, and is said to be *log-concave* if $a_i^2 \ge a_{i-1}a_{i+1}$ for all $1 \le i \le n-1$. Clearly a log-concave sequence of *positive* terms is unimodal. The reader is referred to Stanley's survey [10] for the surprisingly rich variety of methods to show that a sequence is log-concave or unimodal. As noticed by Brenti [2], even though log-concave and unimodality have one-line definitions, to prove the unimodality or log-concavity of a sequence can sometimes be a very difficult task requiring the use of intricate combinatorial constructions or of refined mathematical tools.

Let $\mathfrak{S}_n$ be the set of all permutations of $[n] := \{1,\ldots,n\}$. We say that a permutation $\pi = a_1 a_2 \cdots a_n \in \mathfrak{S}_n$ has a *descent* at i $(1 \le i \le n-1)$ if $a_i > a_{i+1}$. The number of descents of π is called its descent number and is denoted by $\mathrm{d}(\pi)$. A statistic on $\mathfrak{S}_n$ is said to be *Eulerian*, if it is equidistributed with the descent number statistic. Recall that the polynomial

$$A_n(t) = \sum_{\pi \in \mathfrak{S}_n} t^{1+\mathrm{d}(\pi)} = \sum_{k=1}^{n} A(n,k) t^k$$

is called an *Eulerian polynomial*. It is well-known that the *Eulerian numbers* $A(n,k)$ $(1 \le k \le n)$ form a unimodal sequence, of which several proofs have been published: such

[1] guo@math.univ-lyon1.fr, http://math.univ-lyon1.fr/~guo
[2] zeng@math.univ-lyon1.fr, http://math.univ-lyon1.fr/~zeng

as the analytical one by showing that the polynomial $A_n(t)$ has only real zeros [3, p. 294], by induction based on the recurrence relation of $A(n,k)$ (see [9]), or by combinatorial techniques (see [7, 11]).

Let $\mathcal{I}_n$ be the set of all involutions in $\mathfrak{S}_n$ and $\mathcal{J}_n$ the set of all fixed-point free involutions in $\mathfrak{S}_n$. Define

$$I_n(t) = \sum_{\pi \in \mathcal{I}_n} t^{\mathrm{d}(\pi)} = \sum_{k=0}^{n-1} I_{n,k} t^k,$$

$$J_n(t) = \sum_{\pi \in \mathcal{J}_n} t^{\mathrm{d}(\pi)} = \sum_{k=0}^{n-1} J_{n,k} t^k.$$

The first values of these polynomials are given in Table 1.

Table 1: The polynomials $I_n(t)$ and $J_n(t)$ for $n \le 6$.

n	$I_n(t)$	$J_n(t)$
1	1	0
2	$1+t$	t
3	$1+2t+t^2$	0
4	$1+4t+4t^2+t^3$	$t+t^2+t^3$
5	$1+6t+12t^2+6t^3+t^4$	0
6	$1+9t+28t^2+28t^3+9t^4+t^5$	$t+3t^2+7t^3+3t^4+t^5$

As one may notice from Table 1 that the coefficients of $I_n(t)$ and $J_n(t)$ are *symmetric* and *unimodal* for $1 \le n \le 6$. Actually, the symmetries had been conjectured by Dumont and were first proved by Strehl [12]. Recently, Brenti (see [5]) conjectured that the coefficients of the polynomial $I_n(t)$ are *log-concave* and Dukes [5] has obtained some partial results on the unimodality of the coefficients of $I_n(t)$ and $J_{2n}(t)$. Note that, in contrast to Eulerian polynomials $A_n(t)$, the polynomials $I_n(t)$ and $J_{2n}(t)$ may have *non-real zeros*.

In this paper we will prove that for $n \ge 1$, the two sequences $I_{n,0}, I_{n,1}, \ldots, I_{n,n-1}$ and $J_{2n,1}, J_{2n,2}, \ldots, J_{2n,2n-1}$ are unimodal. Our starting point is the known generating functions of polynomials $I_n(t)$ and $J_n(t)$:

$$\sum_{n=0}^{\infty} I_n(t) \frac{u^n}{(1-t)^{n+1}} = \sum_{r=0}^{\infty} \frac{t^r}{(1-u)^{r+1}(1-u^2)^{r(r+1)/2}}, \tag{1.1}$$

$$\sum_{n=0}^{\infty} J_n(t) \frac{u^n}{(1-t)^{n+1}} = \sum_{r=0}^{\infty} \frac{t^r}{(1-u^2)^{r(r+1)/2}}, \tag{1.2}$$

which have been obtained by Désarménien and Foata [4] and Gessel and Reutenauer [8] using different methods. We first derive linear recurrence formulas for $I_{n,k}$ and $J_{2n,k}$ in the next section and then prove the unimodality by induction in Section 3. We end this paper with further conjectures beyond the unimodality of the two sequences $I_{n,k}$ and $J_{2n,k}$.

2 Linear recurrence formulas for $I_{n,k}$ and $J_{2n,k}$

Since the recurrence formula for the numbers $I_{n,k}$ is a little more complicated than $J_{2n,k}$, we shall first prove it for the latter.

Theorem 2.1. *For $n \geq 2$ and $k \geq 0$, the numbers $J_{2n,k}$ satisfy the following recurrence formula:*

$$\begin{aligned} 2nJ_{2n,k} &= [k(k+1)+2n-2]J_{2n-2,k} + 2[(k-1)(2n-k-1)+1]J_{2n-2,k-1} \\ &\quad + [(2n-k)(2n-k+1)+2n-2]J_{2n-2,k-2}. \end{aligned} \tag{2.1}$$

Here and in what follows $J_{2n,k} = 0$ if $k < 0$.

Proof. Equating the coefficients of u^{2n} in (1.2), we obtain

$$\frac{J_{2n}(t)}{(1-t)^{2n+1}} = \sum_{r=0}^{\infty} \binom{r(r+1)/2+n-1}{n} t^r. \tag{2.2}$$

Since

$$\binom{r(r+1)/2+n-1}{n} = \frac{r(r-1)/2+r+n-1}{n} \binom{r(r+1)/2+n-2}{n-1},$$

it follows from (2.2) that

$$\frac{J_{2n}(t)}{(1-t)^{2n+1}} = \frac{t^2}{2n} \left(\frac{J_{2n-2}(t)}{(1-t)^{2n-1}}\right)'' + \frac{t}{n} \left(\frac{J_{2n-2}(t)}{(1-t)^{2n-1}}\right)' + \frac{n-1}{n} \frac{J_{2n-2}(t)}{(1-t)^{2n-1}},$$

or

$$\begin{aligned} J_{2n}(t) &= \frac{t^2(1-t)^2}{2n} J''_{2n-2}(t) + \left[\frac{(2n-1)t^2(1-t)}{n} + \frac{t(1-t)^2}{n}\right] J'_{2n-2}(t) \\ &\quad + \left[(2n-1)t^2 + \frac{(2n-1)(1-t)t}{n} + \frac{(n-1)(1-t)^2}{n}\right] J_{2n-2}(t) \\ &= \frac{t^4-2t^3+t^2}{2n} J''_{2n-2}(t) + \left[\frac{(2-2n)t^3}{n} + \frac{(2n-3)t^2}{n} + \frac{t}{n}\right] J'_{2n-2}(t) \\ &\quad + \left[(2n-2)t^2 + \frac{t}{n} + \frac{n-1}{n}\right] J_{2n-2}(t). \end{aligned} \tag{2.3}$$

Equating the coefficients of t^n in (2.3) yields

$$\begin{aligned} J_{2n,k} &= \frac{(k-2)(k-3)}{2n} J_{2n-2,k-2} - \frac{(k-1)(k-2)}{n} J_{2n-2,k-1} + \frac{k(k-1)}{2n} J_{2n-2,k} \\ &\quad + \frac{(2-2n)(k-2)}{n} J_{2n-2,k-2} + \frac{(2n-3)(k-1)}{n} J_{2n-2,k-1} + \frac{k}{n} J_{2n-2,k} \\ &\quad + (2n-2)J_{2n-2,k-2} + \frac{1}{n} J_{2n-2,k-1} + \frac{n-1}{n} J_{2n-2,k}. \end{aligned}$$

After simplification, we obtain (2.1). ∎

Theorem 2.2. *For $n \geq 3$ and $k \geq 0$, the numbers $I_{n,k}$ satisfy the following recurrence formula:*

$$\begin{aligned} nI_{n,k} &= (k+1)I_{n-1,k} + (n-k)I_{n-1,k-1} + [(k+1)^2 + n - 2]I_{n-2,k} \\ &\quad + [2k(n-k-1) - n + 3]I_{n-2,k-1} + [(n-k)^2 + n - 2]I_{n-2,k-2}. \end{aligned} \tag{2.4}$$

Here and in what follows $I_{n,k} = 0$ if $k < 0$.

Proof. Extracting the coefficients of u^{2n} in (1.1), we obtain

$$\frac{I_n(t)}{(1-t)^{n+1}} = \sum_{r=0}^{\infty} t^r \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{r(r+1)/2 + k - 1}{k} \binom{r + n - 2k}{n - 2k}. \tag{2.5}$$

Let

$$T(n,k) := \binom{x + k - 1}{k} \binom{y + n - 2k}{n - 2k},$$

and

$$s(n) := \sum_{k=0}^{\lfloor n/2 \rfloor} T(n,k).$$

Applying Zeilberger's algorithm, the Maple package `ZeilbergerRecurrence(T,n,k,s,0..n)` gives

$$(2x + y + n + 1)s(n) + (y+1)s(n+1) - (n+2)s(n+2) = 0, \tag{2.6}$$

i.e.,

$$s(n) = \frac{y+1}{n} s(n-1) + \frac{2x + y + n - 1}{n} s(n-2).$$

When $x = r(r+1)/2$ and $y = r$, we get

$$s(n) = \frac{r+1}{n} s(n-1) + \frac{r(r-1) + 3r + n - 1}{n} s(n-2). \tag{2.7}$$

Now, from (2.5) and (2.7) follows the identity

$$\begin{aligned} \frac{nI_n(t)}{(1-t)^{n+1}} &= t \left(\frac{I_{n-1}(t)}{(1-t)^n} \right)' + \frac{I_{n-1}(t)}{(1-t)^n} + t^2 \left(\frac{I_{n-2}(t)}{(1-t)^{n-1}} \right)'' + 3t \left(\frac{I_{n-2}(t)}{(1-t)^{n-1}} \right)' \\ &\quad + (n-1) \frac{I_{n-2}(t)}{(1-t)^{n-1}}, \end{aligned}$$

or

$$\begin{aligned} nI_n(t) &= (t - t^2)I'_{n-1}(t) + [1 + (n-1)t]I_{n-1}(t) + t^2(1-t)^2 I''_{n-2}(t) \\ &\quad + t(1-t)[3 + (2n-5)t]I'_{n-2}(t) + (n-1)[1 + t + (n-2)t^2]I_{n-2}(t). \end{aligned} \tag{2.8}$$

Comparing the coefficients of t^k in both sides of (2.8), we obtain

$$\begin{aligned} nI_{n,k} &= kI_{n-1,k} - (k-1)I_{n-1,k-1} + I_{n-1,k} + (n-1)I_{n-1,k-1} \\ &\quad + k(k-1)I_{n-2,k} - 2(k-1)(k-2)I_{n-2,k-1} + (k-2)(k-3)I_{n-2,k-2} \\ &\quad + 3kI_{n-2,k} + (2n-8)(k-1)I_{n-2,k-1} - (2n-5)(k-2)I_{n-2,k-2} \\ &\quad + (n-1)I_{n-2,k} + (n-1)I_{n-2,k-1} + (n-1)(n-2)I_{n-2,k-2}, \end{aligned}$$

which, after simplification, equals the right-hand side of (2.4). ∎

Remark. The recurrence formula (2.6) can also be proved by hand as follows. It is easy to see that the generating function of $s(n)$ is

$$\sum_{n=0}^{\infty} s(n)u^n = (1-u^2)^{-x}(1-u)^{-y-1}. \tag{2.9}$$

Differentiating (2.9) with respect to u implies that

$$\sum_{n=0}^{\infty} ns(n)u^{n-1} = \left(\frac{2ux}{1-u^2} + \frac{y+1}{1-u}\right)(1-u^2)^{-x}(1-u)^{-y-1},$$

consequently,

$$\begin{aligned} (1-u^2)\sum_{n=0}^{\infty} ns(n)u^{n-1} &= [(2x+y+1)u+y+1](1-u^2)^{-x}(1-u)^{-y-1} \\ &= [(2x+y+1)u+y+1]\sum_{n=0}^{\infty} s(n)u^n. \end{aligned} \tag{2.10}$$

Comparing the coefficients of u^{n+1} in both sides of (2.10), we obtain

$$(n+2)s(n+2) - ns(n) = (2x+y+1)s(n) + (y+1)s(n+1),$$

which is equivalent to (2.6).

Note that the right-hand side of (2.1) (respectively, (2.4)) is invariant under the substitution $k \to 2n-k$ (respectively, $k \to n-1-k$), provided that the sequence $I_{n-1,k}$ (respectively, $J_{2n-2,k}$) is symmetric. Thus, by induction we derive immediately the symmetry properties of $J_{2n,k}$ and $I_{n,k}$ (see [4, 8, 12]).

Corollary 2.3. *For $n, k \in \mathbb{N}$, we have*

$$I_{n,k} = I_{n,n-1-k}, \quad J_{2n,k} = J_{2n,2n-k}.$$

It would be interesting to find a combinatorial proof of the recurrence formulas (2.1) and (2.4), since such a proof could hopefully lead to a combinatorial proof of the unimodality of these two sequences.

3 Unimodality of the sequences $I_{n,k}$ and $J_{2n,k}$

The following observation is crucial in our inductive proof of the unimodality of the sequences $I_{n,k}$ $(0 \le k \le n-1)$ and $J_{2n,k}$ $(1 \le k \le 2n-1)$.

Lemma 3.1. *Let $x_0, x_1, \ldots, x_n$ and $a_0, a_1, \ldots, a_n$ be real numbers such that $x_0 \ge x_1 \ge \cdots \ge x_n \ge 0$ and $a_0 + a_1 + \cdots + a_k \ge 0$ for all $k = 0, 1, \ldots, n$. Then*

$$\sum_{i=0}^{n} a_i x_i \ge 0.$$

Indeed, the above inequality follows from the identity:

$$\sum_{i=0}^{n} a_i x_i = \sum_{k=0}^{n} (x_k - x_{k+1})(a_0 + a_1 + \cdots + a_k),$$

where $x_{n+1} = 0$.

Theorem 3.2. *The sequence $J_{2n,1}, J_{2n,2}, \ldots, J_{2n,2n-1}$ is unimodal.*

Proof. By the symmetry of $J_{2n,k}$, it is enough to show that $J_{2n,k} \ge J_{2n,k-1}$ for all $2 \le k \le n$. We proceed by induction on n. The $n = 2$ case is clear from Table 1. Suppose the sequence $J_{2n-2,k}$ is unimodal in k. By Theorem 2.1, one has

$$2n(J_{2n,k} - J_{2n,k-1}) = A_0 J_{2n-2,k} + A_1 J_{2n-2,k-1} + A_2 J_{2n-2,k-2} + A_3 J_{2n-2,k-3}, \tag{3.1}$$

where

$$A_0 = k^2 + k + 2n - 2, \quad A_1 = 4nk - 3k^2 - 6n + k + 6,$$
$$A_2 = 3k^2 + 4n^2 - 8nk - 5k + 12n - 4, \quad A_3 = 3k - k^2 + 4nk - 4n^2 - 8n.$$

We have the following two cases:

- If $2 \le k \le n-1$, then

$$J_{2n-2,k} \ge J_{2n-2,k-1} \ge J_{2n-2,k-2} \ge J_{2n-2,k-3}$$

  by the induction hypothesis, and clearly

$$A_0 \ge 0, \quad A_0 + A_1 = 2(k-1)(2n-k) + 4 \ge 0,$$
$$A_0 + A_1 + A_2 = (2n-k)^2 - 3k + 8n \ge 0, \quad A_0 + A_1 + A_2 + A_3 = 0.$$

  Therefore, by Lemma 3.1, we have

$$J_{2n,k} - J_{2n,k-1} \ge 0.$$

- If $k = n$, then
$$J_{2n-2,n-1} \geq J_{2n-2,n} = J_{2n-2,n-2} \geq J_{2n-2,n-3}$$
by symmetry and the induction hypothesis. In this case, we have $A_1 = (n-2)(n-3) \geq 0$ and thus the corresponding condition of Lemma 3.1 is satisfied. Therefore, we have
$$J_{2n,n} - J_{2n,n-1} \geq 0.$$

This completes the proof. ∎

Theorem 3.3. *The sequence $I_{n,0}, I_{n,1}, \ldots, I_{n,n-1}$ is unimodal.*

Proof. By the symmetry of $I_{n,k}$, it suffices to show that $I_{n,k} \geq I_{n,k-1}$ for all $1 \leq k \leq (n-1)/2$. From Table 1, it is clear that the sequences $I_{n,k}$ are unimodal in k for $1 \leq n \leq 6$.

Now suppose $n \geq 7$ and the sequences $I_{n-1,k}$ and $I_{n-2,k}$ are unimodal in k. Replacing k by $k-1$ in (2.4), we obtain

$$\begin{aligned} nI_{n,k-1} &= kI_{n-1,k-1} + (n-k+1)I_{n-1,k-2} + (k^2+n-2)I_{n-2,k-1} \\ &\quad + [2(k-1)(n-k) - n + 3]I_{n-2,k-2} + [(n-k+1)^2 + n - 2]I_{n-2,k-3}. \end{aligned} \tag{3.2}$$

Combining (2.4) and (3.2) yields

$$\begin{aligned} n(I_{n,k} - I_{n,k-1}) &= B_0 I_{n-1,k} + B_1 I_{n-1,k-1} + B_2 I_{n-1,k-2} \\ &\quad + C_0 I_{n-2,k} + C_1 I_{n-2,k-1} + C_2 I_{n-2,k-2} + C_3 I_{n-2,k-3}, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} &B_0 = k+1, \quad B_1 = n - 2k, \quad B_2 = -(n-k+1), \\ &C_0 = (k+1)^2 + n - 2, \quad C_1 = 2nk - 3k^2 - 2k - 2n + 5, \\ &C_2 = n^2 - 4nk + 3k^2 + 4n - 2k - 5, \quad C_3 = -(n-k+1)^2 - n + 2. \end{aligned}$$

Notice that $I_{n-1,k} \geq I_{n-1,k-1} \geq I_{n-1,k-2}$ for $1 \leq k \leq (n-1)/2$. By Lemma 3.1, we have

$$B_0 I_{n-1,k} + B_1 I_{n-1,k-1} + B_2 I_{n-1,k-2} \geq 0. \tag{3.4}$$

It remains to show that

$$C_0 I_{n-2,k} + C_1 I_{n-2,k-1} + C_2 I_{n-2,k-2} + C_3 I_{n-2,k-3} \geq 0, \quad \forall\, 1 \leq k \leq (n-1)/2. \tag{3.5}$$

We need to consider the following two cases:

- If $1 \leq k \leq (n-2)/2$, then
$$I_{n-2,k} \geq I_{n-2,k-1} \geq I_{n-2,k-2} \geq I_{n-2,k-3}$$
by the induction hypothesis, and
$$\begin{aligned} &C_0 = (k+1)^2 + n - 2 \geq 0, \quad C_0 + C_1 = (2k-1)(n-k-1) + k + 3 \geq 0, \\ &C_0 + C_1 + C_2 = (n-k+1)^2 + n - 2 \geq 0, \quad C_0 + C_1 + C_2 + C_3 = 0. \end{aligned}$$

- If $k = (n-1)/2$, then by symmetry and the induction hypothesis,

$$I_{n-2,k-1} \geq I_{n-2,k} = I_{n-2,k-2} \geq I_{n-2,k-3}.$$

In this case, we have $C_1 = (n-3)(n-7)/4 \geq 0$ for $n \geq 7$.

Therefore, by Lemma 3.1 the inequality (3.5) holds. It follows from (3.3)–(3.5) that

$$I_{n,k} - I_{n,k-1} \geq 0, \quad \forall\, 1 \leq k \leq (n-1)/2.$$

This completes the proof. ∎

4 Further remarks and open problems

Since $I_{n,k} = I_{n,n-1-k}$, we can rewrite $I_n(t)$ as follows:

$$I_n(t) = \sum_{k=0}^{n-1} I_{n,k} t^k = \begin{cases} \displaystyle\sum_{k=0}^{n/2-1} I_{n,k} t^k (1 + t^{n-2k-1}), & \text{if } n \text{ is even,} \\ \displaystyle I_{n,(n-1)/2} t^{(n-1)/2} + \sum_{k=0}^{(n-3)/2} I_{n,k} t^k (1 + t^{n-2k-1}), & \text{if } n \text{ is odd.} \end{cases}$$

Applying the well-known formula

$$x^n + y^n = \sum_{j=0}^{\lfloor n/2 \rfloor} (-1)^j \frac{n}{n-j} \binom{n-j}{j} (xy)^j (x+y)^{n-2j},$$

we obtain

$$I_n(t) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k} t^k (1+t)^{n-2k-1}, \tag{4.1}$$

where

$$a_{n,k} = \begin{cases} \displaystyle\sum_{j=0}^{k} (-1)^{k-j} \frac{n-2j-1}{n-k-j-1} \binom{n-k-j-1}{k-j} I_{n,j}, & \text{if } 2k+1 < n, \\ \displaystyle I_{n,k} + \sum_{j=0}^{k-1} (-1)^{k-j} \frac{n-2j-1}{n-k-j-1} \binom{n-k-j-1}{k-j} I_{n,j}, & \text{if } 2k+1 = n. \end{cases}$$

The first values of $a_{n,k}$ are given in Table 2, which seems to suggest the following conjecture.

Conjecture 4.1. *For $n \geq 1$ and $k \geq 0$, the coefficients $a_{n,k}$ are nonnegative integers.*

Table 2: Values of $a_{n,k}$ for $n \le 16$ and $0 \le k \le \lfloor (n-1)/2 \rfloor$.

$k \setminus n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
0	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
1			0	1	2	4	6	9	12	16	20	25	30	36	42	49
2					2	6	18	39	79	141	239	379	579	849	1211	1680
3							0	18	78	272	722	1716	3626	7160	13206	23263
4									20	124	668	2560	8360	23536	59824	139457
5											32	700	4800	24160	95680	325572
6													440	5480	44632	257964
7															2176	44376

Since the coefficients of $t^k(1+t)^{n-2k-1}$ are symmetric and unimodal with center of symmetry at $(n-1)/2$, Conjecture 4.1 is stronger than the fact that the coefficients of $I_n(t)$ are symmetric and unimodal. A more interesting question is to give a combinatorial interpretation of $a_{n,k}$. Note that the Eulerian polynomials can be written as

$$A_n(t) = \sum_{k=1}^{\lfloor (n+1)/2 \rfloor} c_{n,k} t^k (1+t)^{n-2k+1},$$

where $c_{n,k}$ is the number of increasing binary trees on $[n]$ with k leaves and no vertices having left children only (see [1, 6, 7]).

We now proceed to derive a recurrence relation for $a_{n,k}$. Set $x = x(t) = t/(1+t)^2$ and

$$P_n(x) = \sum_{k=0}^{\lfloor (n-1)/2 \rfloor} a_{n,k} x^k.$$

Then we can rewrite (4.1) as

$$I_n(t) = (1+t)^{n-1} P_n(x). \tag{4.2}$$

Differentiating (4.2) with respect to t we get

$$I_n'(t) = (n-1)(1+t)^{n-2} P_n(x) + (1+t)^{n-1} P_n'(x) x'(t), \tag{4.3}$$

$$\begin{aligned} I_n''(t) = {} & (n-1)(n-2)(1+t)^{n-3} P_n(x) + 2(n-1)(1+t)^{n-2} P_n'(x) x'(t) \\ & + (1+t)^{n-1} P_n''(x)(x'(t))^2 + (1+t)^{n-1} P_n'(x) x''(t), \end{aligned} \tag{4.4}$$

$$x'(t) = \frac{1-t}{(1+t)^3}, \quad x''(t) = \frac{2t-4}{(1+t)^4}. \tag{4.5}$$

Substituting (4.2)–(4.5) into (2.8), we obtain

$$\begin{aligned}
&n(1+t)^{n-1}P_n(x)\\
&\quad = [1+(2n-2)t+t^2](1+t)^{n-3}P_{n-1}(x)+t(1-t)^2(1+t)^{n-5}P'_{n-1}(x)\\
&\qquad +[-(t^2+14t+1)(1-t)^2+(1+6t-18t^2+6t^3+t^4)n+4t^2n^2](1+t)^{n-5}P_{n-2}(x)\\
&\qquad +[3t(t^2-4t+1)(1-t)^2+4t^2(1-t)^2n](1+t)^{n-7}P'_{n-2}(x)\\
&\qquad +t^2(1-t)^4(1+t)^{n-9}P''_{n-2}(x).
\end{aligned} \tag{4.6}$$

Dividing the two sides of (4.6) by $(1+t)^{n-1}$ and noticing that $t/(1+t)^2=x$, after a little manipulation we get

$$\begin{aligned}
nP_n(x) &= [1+(2n-4)x]P_{n-1}(x)+(x-4x^2)P'_{n-1}(x)\\
&\quad +[(n-1)+(2n-8)x+4(n-3)(n-4)x^2]P_{n-2}(x)\\
&\quad +[3x+(4n-30)x^2+(72-16n)x^3]P'_{n-2}(x)+(x^2-8x^3+16x^4)P''_{n-2}(x).
\end{aligned}$$

Extracting the coefficients of x^k yields

$$\begin{aligned}
na_{n,k} &= a_{n-1,k}+(2n-4)a_{n-1,k-1}+ka_{n-1,k}-4(k-1)a_{n-1,k-1}\\
&\quad +(n-1)a_{n-2,k}+(2n-8)a_{n-2,k-1}+4(n-3)(n-4)a_{n-2,k-2}\\
&\quad +3ka_{n-2,k}+(4n-30)(k-1)a_{n-2,k-1}+(72-16n)(k-2)a_{n-2,k-2}\\
&\quad +k(k-1)a_{n-2,k}-8(k-1)(k-2)a_{n-2,k-1}+16(k-2)(k-3)a_{n-2,k-2}.
\end{aligned}$$

After simplification, we obtain the following recurrence formula for $a_{n,k}$.

Theorem 4.2. *For $n \geq 3$ and $k \geq 0$, there holds*

$$\begin{aligned}
na_{n,k} &= (k+1)a_{n-1,k}+(2n-4k)a_{n-1,k-1}+[k(k+2)+n-1]a_{n-2,k}\\
&\quad +[(k-1)(4n-8k-14)+2n-8]a_{n-2,k-1}+4(n-2k)(n-2k+1)a_{n-2,k-2},
\end{aligned} \tag{4.7}$$

where $a_{n,k}=0$ if $k<0$ or $k>(n-1)/2$.

Note that, if $n \geq 2k+3$, then

$$(k-1)(4n-8k-14)+2n-8>0 \quad \text{for any } k \geq 1,$$

and so are the other coefficients in (4.7). Therefore, Conjecture 4.1 would be proved if one can show that $a_{2n+1,n} \geq 0$ and $a_{2n+2,n} \geq 0$.

Finally, from (4.1) it is easy to see that

$$a_{2n+1,n} = (-1)^n I_{2n+1}(-1) = \sum_{k=0}^{2n} (-1)^{n-k} I_{2n+1,k},$$

$$a_{2n+2,n} = (-1)^n I'_{2n+2}(-1) = \sum_{k=1}^{2n+1} (-1)^{n+1-k} k I_{2n+2,k}.$$

Thus, Conjecture 4.1 is equivalent to the *nonnegativity* of the above two alternating sums.

Since $J_{2n,k} = J_{2n,2n-k}$, in the same manner as $I_n(t)$ we obtain

$$J_{2n}(t) = \sum_{k=1}^{n} b_{2n,k} t^k (1+t)^{2n-2k},$$

where

$$b_{2n,k} = \begin{cases} \displaystyle\sum_{j=1}^{k} (-1)^{k-j} \frac{2n-2j}{2n-k-j} \binom{2n-k-j}{k-j} J_{2n,j}, & \text{if } k < n, \\ \displaystyle J_{2n,k} + \sum_{j=1}^{k-1} (-1)^{k-j} \frac{2n-2j}{2n-k-j} \binom{2n-k-j}{k-j} J_{2n,j}, & \text{if } k = n. \end{cases}$$

Now, it follows from (2.2) that

$$J_{2n,k} = \sum_{i=0}^{k} (-1)^{k-i} \binom{2n+1}{k-i} \binom{i(i+1)/2+n-1}{i(i+1)/2-1}$$

is a polynomial in n of degree $d := k(k+1)/2 - 1$ with leading coefficient $1/d!$, and so is $b_{2n,k}$. Thus, we have $\lim_{n\to+\infty} b_{2n,k} = +\infty$ for any fixed $k > 1$.

The first values of $b_{2n,k}$ are given in Table 3, which seems to suggest

Conjecture 4.3. *For $n \geq 9$ and $k \geq 1$, the coefficients $b_{2n,k}$ are nonnegative integers.*

Similarly to the proof of Theorem 4.2, we can prove the following result.

Theorem 4.4. *For $n \geq 2$ and $k \geq 1$, there holds*

$$\begin{aligned} 2n b_{2n,k} = {} & [k(k+1) + 2n - 2] b_{2n-2,k} + [2 + 2(k-1)(4n-4k-3)] b_{2n-2,k-1} \\ & + 8(n-k+1)(2n-2k+1) b_{2n-2,k-2}. \end{aligned}$$

where $b_{2n,k} = 0$ if $k < 1$ or $k > n$.

Theorem 4.4 allows us to reduce the verification of Conjecture 4.3 to the boundary case $b_{2n,n} \geq 0$ for $n \geq 9$.

Acknowledgment. The second author was supported by EC's IHRP Programme, within Research Training Network "Algebraic Combinatorics in Europe," grant HPRN-CT-2001-00272.

Table 3: Values of $b_{2n,k}$ for $2n \le 24$ and $1 \le k \le n$.

$k \setminus 2n$	2	4	6	8	10	12	14	16	18	20	22	24
1	1	1	1	1	1	1	1	1	1	1	1	1
2		-1	-1	0	2	5	9	14	20	27	35	44
3			3	12	36	91	201	399	728	1242	2007	3102
4				-7	-10	91	652	2593	7902	20401	46852	98494
5					25	219	1710	10532	50165	194139	639968	1861215
6						-65	249	11319	122571	841038	4377636	18747924
7							283	6586	135545	1737505	15219292	101116704
8								-583	33188	1372734	24412940	277963127
9									4417	379029	16488999	367507439
10										1791	3350211	203698690
11											133107	36903128
12												761785

References

[1] P. Brändén, Sign-graded posets, unimodality of W-polynomials and the Charney-Davis conjecture, Electron. J. Combin. **11** (2) (2004/05), #R9.

[2] F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update, Jerusalem combinatorics'93, pp. 71–89, Contemp. Math., **178**, Amer. Math. Soc., Providence, RI, 1994.

[3] L. Comtet, Advanced Combinatorics, Reidel, Dordrecht/Boston, 1974.

[4] J. Désarménien and D. Foata, Fonctions symétriques et séries hypergéométriques basiques multivariées, Bull. Soc. Math. France **113** (1985), 3–22.

[5] W. M. B. Dukes, Permutation statistics on involutions, European J. Combin., to appear.

[6] D. Foata and V. Strehl, Euler numbers and variations of permutations, Atti dei Convegni Lincei, **17** Tomo I, 1976, pp. 119–131.

[7] V. Gasharov, On the Neggers-Stanley conjecture and the Eulerian polynomials, J. Combin. Theory Ser. A **82** (1998), 134–146.

[8] I. M. Gessel and C. Reutenauer, Counting permutations with given cycle structure and descent set, J. Combin. Theory Ser. A **64** (1993), 189–215.

[9] D. C. Kurtz, A note on concavity properties of triangular arrays of numbers, J. Combin. Theory Ser. A **13** (1972), 135–139.

[10] R. P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, Ann. New York Acad. Sci. **576** (1989), 500–535.

[11] J. R. Stembridge, Eulerian numbers, tableaux, and the Betti numbers of a toric variety, Discrete Math. **99** (1992), 307–320.

[12] V. Strehl, Symmetric Eulerian distributions for involutions, Séminair Lotharingien Combinatoire **1**, Strasbourg 1980, Publications de l'I.R.M.A. 140/S-02, Strasbourg, 1981.